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EVALUATION OF THE PROBABILITY INTEGRAL

BY FRANK GILMAN

A SIMPLE and convenient formula for calculating the numerical value of the definite integral

$$\int_0^x e^{-x^2} dx$$

has long seemed to me a desideratum. The three series commonly used for computing this integral are not convenient in practice, except when the argument x is either very small or very large. It is believed that the formulas here given will be found equally convenient for any value of x from 0 to 3, and that they will, in general, give the integral correct to five figures. It is true that some of the old tables give this integral to eleven, and some even to fourteen decimals; but so great a degree of accuracy appears to be unnecessary. Besides, the method here given can be extended so as to secure any degree of accuracy desired.

The integral

$$\int_0^x e^{-x^2} dx$$

is of great importance in many branches of applied mathematics. It is used in discussing the effect of refraction,* in investigating the secular cooling of the earth † and in discussions in regard to the conduction of heat, as found in Fourier's work, *Théorie analytique de la chaleur*.

* See Chauvenet's *Astronomy*, vol. 1.

† See Thompson and Tait's *Natural Philosophy*, vol. 1, p. 717.

It is of the greatest importance in the theory of probability, or probability of errors, and on this account, when written in the form

$$\int_x^\infty e^{-x^2} dx = \operatorname{erf} x$$

it is sometimes called the error function of x , while

$$\int_0^x e^{-x^2} dx = \operatorname{erfc} x$$

is called the error function complement of x . The former of these is designated by the notation $\operatorname{erf} x$, and the latter by $\operatorname{erfc} x$. The two are so related that

$$\operatorname{erf} x + \operatorname{erfc} x = \int_0^\infty e^{-x^2} dx,$$

or

$$\operatorname{erf} x + \operatorname{erfc} x = \frac{1}{2}\sqrt{\pi}.$$

We shall give formulas for computing directly $\operatorname{erfc} x$, since, this being known, $\operatorname{erf} x$ can be immediately deduced.

Besides the applications of $\operatorname{erf} x$ above mentioned, it is of great use in the integration of many differential equations (Kramp says an infinite number) and Glaisher in the *Philosophical Magazine* for 1871,* has given a list of some of the definite integrals which can be expressed in terms of it.

Glaisher says that from its uses in physics, $\operatorname{erf} x$ may fairly claim at present to rank in importance next to the trigonometrical and logarithmic functions.

The following is the formula proposed for computing $\operatorname{erfc} x$ for values of x from 0 to 1:

$$\int_0^x e^{-x^2} dx = A + ax + bx^3 + cx^5. \quad (1)$$

When $x > 1$ we may write

$$\int_0^x e^{-x^2} dx = A + \frac{a}{x} + \frac{b}{x^3} + \frac{c}{x^5}. \quad (2)$$

* Vol. 42, pp. 294, 421.

The following are the values of the constants A , a , b , and c , corresponding to assigned limits of x .

Limits of x	A	Log a	Log b	Log c
0 to 0.40	0	0	9.5221346 n	8.9674202
0.40 " .55	.0000225	9.9995025	9.5148674 n	8.9090072
.55 " .70	.002127	9.9966907	9.4956431 n	8.8260698
.70 " .85	.009435	9.9882445	9.4598519 n	8.7203776
.85 " 1.00	.024000	9.9739908	9.4159907 n	8.6187244
1.00 " 1.15	.960956	8.8864253 n	9.2954356 n	8.7802668
1.15 " 1.30	.871946	9.0553760	9.5600979 n	9.0974426
1.30 " 1.60	.809877	9.4213859	9.7244149 n	9.3195134
1.60 " 2.00	.831292	9.2883208	9.5942662 n	8.9499540
2.00 " 2.40	.883119	7.5394525	8.9915539	9.6821021 n
2.40 " 3.00	.893330	8.6048101 n	9.3971250	9.8547853 n

The following was the method of calculating these quantities. Assume

$$a' + b'x^2 + c'x^4 = e^{-x^2}.$$

Substitute in this equation certain values of x , and from the condition equations thus obtained, form the normal equations, according to the method of least squares. The solution of these equations will give the values of a' , b' , and c' . Then by integration we obtain

$$\int_0^x e^{-x^2} dx = A + a'x + \frac{b'}{3} x^3 + \frac{c'}{5} x^5,$$

in which A is a constant, $a = a'$, $b = \frac{1}{3}b'$, $c = \frac{1}{5}c'$.

When $x > 1$ we write

$$\frac{a'}{x^2} + \frac{b'}{x^4} + \frac{c'}{x^6} = e^{-x^2}$$

and proceed in the same way, remembering that $a = -a'$, $b = -\frac{1}{3}b'$, $c = -\frac{1}{5}c'$.

In order to explain more fully the method in detail, let us find the coeffi-

cients A , a , b , and c corresponding to the limits of x : 1.60 to 2.00. Substitute in the equation

$$\frac{a'}{x^2} + \frac{b'}{x^4} + \frac{c'}{x^6} = e^{-x^2}$$

the following successive values of x : 1.6, 1.65, 1.7, 1.75, 1.8, 1.85, 1.9, 1.95, 2.00 and write the 9 equations of condition:

$$\begin{aligned} 0.39062496a' + 0.15258786b' + 0.05960463c' &= 0.07730473, \\ .36730950a' + .13491630b' + .04955605c' &= .06571026, \\ .34602080a' + .11973039b' + .04142920c' &= .05557621, \\ .32653068a' + .10662229b' + .03481545c' &= .04677061, \\ .30864200a' + .09525987b' + .02940120c' &= .03916391, \\ .29218412a' + .08537155b' + .02494421c' &= .03263074, \\ .27700833a' + .07673361b' + .02125585c' &= .02705186, \\ .26298491a' + .06916105b' + .01818831c' &= .02231492, \\ .25000000a' + .06250000b' + .01562500c' &= .01831564, \end{aligned}$$

where the second member is in each case the value of e^{-x^2} .

From these condition equations the normal equations are formed in the usual way, and are as follows:

$$\begin{aligned} 0.90288292a' + 0.29481990b' + 0.09812945c' &= 0.12839856 \\ 0.29481990a' + 0.09812945b' + 0.03324910c' &= 0.04358243 \\ 0.09812945a' + 0.03324910b' + 0.01145041c' &= 0.01502736 \end{aligned}$$

Solving these equations we find $\log a' = 9.2883208n$, $\log b' = 0.0713875$, $\log c' = 9.6489240n$; and from the known relations between a' , b' , c' and a , b , c , we have $\log a = 9.2883208$, $\log b = 9.5942662n$, $\log c = 8.9499540$, as given in the table. The constant A is determined from the condition that when $x = 1.6$ formula (2) should give the correct value of

$$\int_0^{1.6} e^{-x^2} dx.$$

A similar method was used in obtaining the other coefficients of the table;

but the number of condition equations was not always the same, and in some cases only 4 were used.

Theoretically it would seem as if a large increase in the number of condition equations would add largely to the accuracy of the results, but in practice I have not always found this to be the case.

In order to illustrate the use of the table let it be required to find the value of

$$\int_{0.45}^{1.61} e^{-x^2} dx.$$

The computation is as follows :

$$\begin{array}{lll} \log a = 9.2883208 & \log b = 9.5942662n & \log c = 8.9499540 \\ \log 1.61 = \frac{0.2068259}{9.0814949} & \log (1.61)^3 = \frac{0.6204777}{8.9737885n} & \log (1.61)^5 = \frac{1.0341295}{7.9158245} \end{array}$$

$$\begin{array}{l} A = 0.831292 \\ ax = 0.120641 \\ cx^5 = \frac{0.008238}{.960171} \\ bx^3 = -\frac{.094143}{.866028} = \int_0^{1.61} e^{-x^2} dx \end{array}$$

$$\begin{array}{lll} \log a = 9.9995025 & \log b = 9.5148674n & \log c = 8.9090072 \\ \log .45 = \frac{9.6532125}{9.6527150} & \log (.45)^3 = \frac{8.9596375}{8.4745049n} & \log (.45)^5 = \frac{8.2660625}{7.1750697} \end{array}$$

$$\begin{array}{l} A = 0.000225 \\ ax = 0.449485 \\ cx^5 = \frac{0.001496}{0.451206} \\ bx^3 = \frac{0.029820}{.421386} = \int_0^{.45} e^{-x^2} dx \end{array} \quad \begin{array}{l} .866028 \\ .421386 \\ .444642 = \int_{0.45}^{1.61} e^{-x^2} dx. \end{array}$$

If we attempt to compute

$$\int_0^{1.61} e^{-x^2} dx$$

by the three series commonly used, namely,

$$x - \frac{x^3}{3} + \frac{1}{1.2} \frac{x^5}{5} - \frac{1}{1.2.3} \frac{x^7}{7} + \frac{1}{1.2.3.4} \frac{x^9}{9} - \cdots,$$

$$e^{-x^2} x \left[1 + \frac{2x^2}{3} + \frac{(2x^2)^2}{3.5} + \frac{(2x^2)^3}{3.5.7} + \frac{(2x^2)^4}{3.5.7.9} + \cdots \right],$$

$$\frac{1}{2}\sqrt{\pi} - \frac{e^{-x^2}}{2x} \left[1 - \frac{1}{2x^2} + \frac{1.3}{(2x^2)^2} - \frac{1.3.5}{(2x^2)^3} + \frac{1.3.5.7}{(2x^2)^4} - \cdots \right],$$

we shall find that it will require 9 terms in each of the first two series to give the integral correct to three figures, while it cannot be obtained correct to three figures by using the third series, no matter how many terms are taken.

As previously stated, it is believed that the formulas which have been given will, in general, give the integral correct to five figures, for the reason that I have made comparisons of integrals computed by these formulas, for values of x from 0 to 2, and at intervals of every 0.02, with the values as given in standard tables. In the hundred comparisons thus made I found no discrepancy greater than one in the fifth figure.

The reason why the method above explained should be expected to give accurate results was explained in an article on the ballistic problem, published in the *ANNALS OF MATHEMATICS* for April, 1905.

It was not deemed advisable to extend the table of coefficients for values of x greater than 3, as the third series commonly used is very convenient for values of x beyond that limit.

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